函数展开成傅里叶级数

1. $f(x)$ 是周期为 2π 的周期函数.

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx+b_n\sin nx.$$

$$\Rightarrow \boxed{\begin{aligned} a_0&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx\\ a_n&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx\\ b_n&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx\end{aligned}}$$

推导过程. $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx+b_n\sin nx$ (级数收敛)

① 两边积分 $\Rightarrow$ $\int_{-\pi}^{\pi}f(x)\,dx=\int_{-\pi}^{\pi}\frac{a_0}{2}\,dx+\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx$

$$=\frac{a_0}{2}x\Big|_{-\pi}^{\pi}+\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}a_n\cos nx\,dx+\int_{-\pi}^{\pi}b_n\sin nx\,dx$$

$$=\pi\cdot a_0+0$$

$$=\pi a_0 \quad\Rightarrow\quad a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

② 两边乘以 $\cos nx$ 再积分

$$\Rightarrow \int_{-\pi}^{\pi}f(x)\cos nx\,dx=\int_{-\pi}^{\pi}\frac{a_0}{2}\cdot\cos nx\,dx+\sum_{k=1}^{\infty}\int_{-\pi}^{\pi}a_k\cos kx\cdot\cos nx\,dx$$

$$+\sum_{k=1}^{\infty}\int_{-\pi}^{\pi}b_k\sin kx\cos nx\,dx$$

$$=0+\int_{-\pi}^{\pi}a_n\cos^2 nx\,dx+0$$

$$=a_n\int_{-\pi}^{\pi}\frac{1+\cos(2nx)}{2}\,dx$$

$$=a_n\cdot\frac{1}{2}\left[x+\frac{1}{2n}\sin(2nx)\right]\Big|_{-\pi}^{\pi}$$

$$= \pi a_n \quad \Rightarrow \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

③ 两边乘以 $\sin nx$ 再积分：

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\sin nx\,dx = \int_{-\pi}^{\pi} b_n \sin^2 nx\,dx$$

$$= b_n \int_{-\pi}^{\pi} \frac{1-\cos(2nx)}{2}\,dx$$

$$= b_n \frac{1}{2}\left(x - \frac{1}{2n}\sin 2nx\right)\Big|_{-\pi}^{\pi}$$

$$= \pi b_n \quad \Rightarrow \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

2. 定理：$f(x)$ 是周期为 2π 的函数，若

① 在一个周期内连续或只有有限个第一类间断点（左右存在）

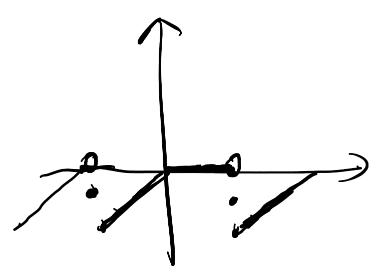
② 在一个周期内有有限个极值点

$\Rightarrow$ x 是连续点时 $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n \sin nx$ 收敛到 $f(x)$

x 是间断点时，级数收敛到 $\frac{1}{2}[f(x^+) + f(x^-)]$

例1. 设 $f(x)$ 是周期为 2π 的函数，在 $[-\pi, \pi]$，

$$f(x) = \begin{cases} x & -\pi \le x < 0 \\ 0 & 0 \le x < \pi \end{cases}$$



解：$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \int_{0}^{\pi} 0\,dx$

$$= \frac{1}{\pi}\frac{1}{2}x^2\Big|_{-\pi}^{0} = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx + 0$$

$$= \frac{1}{\pi}\cdot\left[\frac{1}{n}x\sin nx - \frac{1}{n}\int_{-\pi}^{\pi}\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\cdot\frac{1}{n^2} - \frac{1}{\pi n^2}\cos(-n\pi) = \frac{1}{\pi n^2}\left(1-(-1)^n\right) = \begin{cases} \frac{2}{\pi n^2} & n = 2m+1 \\ 0 & n = 2m \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}x\cos nx + \frac{1}{n}\int_{-\pi}^{0}\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx\right]\Big|_{-\pi}^{0}$$

$$= \frac{1}{\pi n}\left(-\pi\cos(-n\pi)\right) = -\frac{(-1)^n}{n} = \begin{cases} -\frac{1}{n} & n = 2m \\ +\frac{1}{n} & n = 2m+1 \end{cases}$$

$$\Rightarrow\quad f(x) = -\frac{\pi}{4} + \left(\frac{2}{\pi}\cos x + \sin x\right) - \frac{1}{2}\sin 2x + \left(\frac{1}{9\pi}\cos 3x + \frac{1}{3}\sin 3x\right) + \cdots$$

$$= -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right)$$