基本初等函数的导数

笔记下载: www.sudoedu.com/cn

1. 导数的定义: $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

2. $\sin x$ 的导数

$$(\sin x)' = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\sin h\cos x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h} + \lim_{h\to 0}\frac{\sin h}{h}\cdot\cos x$$

$$= \cos x$$

$$\Rightarrow (\sin x)' = \cos x$$

$$\boxed{\begin{aligned}&\sin(A+B) = \sin A\cos B+\sin B\cos A\\ &\cos(A+B) = \cos A\cos B-\sin A\sin B\end{aligned}}$$

$$\lim_{x\to 0}\frac{\cos x-1}{x} = 0$$

$$\lim_{x\to 0}\frac{-2\sin^2\frac{x}{2}}{x} = \lim_{x\to 0}\frac{-2\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\cdot\frac{x}{4}$$

$$= 0$$

3. $$(\cos x)' = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x(\cos h-1)}{h} - \lim_{h\to 0}\frac{\sin x\sin h}{h} = -\sin x$$

$$\Rightarrow (\cos x)' = -\sin x$$

4. $$(x^n)' = \lim_{h\to 0}\frac{(x+h)^n-x^n}{h} = x^n\lim_{h\to 0}\frac{\left(1+\frac{h}{x}\right)^n-1}{h}$$

$n\in\mathbb{N}$

$$= x^n\cdot\lim_{h\to 0}\frac{\frac{\cancel{h}}{x}\left(1+\left(1+\frac{h}{x}\right)+\left(1+\frac{h}{x}\right)^2+\cdots+\left(1+\frac{h}{x}\right)^{n-1}\right)}{\cancel{h}}$$

$$= x^n\cdot\frac{1}{x}\cdot n = nx^{n-1}$$

$$\Rightarrow (x^n)' = nx^{n-1}\quad (n\in\mathbb{N})$$

$$\boxed{\begin{aligned}&1+r+r^2+\cdots+r^{n-1} = \frac{r^n-1}{r-1}\\ &\Rightarrow r^n-1 = (r-1)(1+r+\cdots+r^{n-1})\end{aligned}}$$

5. $$(e^x)' = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0} e^x\cdot\frac{e^h-1}{h} = e^x\lim_{h\to 0}\frac{e^h-1}{h}$$

$$\frac{t=e^h-1}{h=\ln(t+1)}\quad e^x\cdot\lim_{t\to 0}\frac{t}{\ln(1+t)} = e^x\lim_{t\to 0}\frac{1}{\frac{1}{t}\ln(1+t)} = e^x\lim_{t\to 0}\frac{1}{\ln(1+t)^{\frac{1}{t}}}$$

$$= e^x$$

$(a^x)' = a^x$

$$(e^x)' = e^x$$

6. $$(a^x)' = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = a^x\lim_{h\to 0}\frac{a^h-1}{h} \xlongequal[h=\log_a(1+t)]{t=a^h-1} a^x\lim_{t\to 0}\frac{t}{\log_a(1+t)}$$

$$= a^x\cdot\lim_{t\to 0}\frac{t}{\ln(1+t)/\ln a} = a^x\ln a\lim_{t\to 0}\frac{t}{\ln(1+t)}$$

$$= a^x\ln a$$

$$\boxed{\log_a x = \frac{\log_b x}{\log_b a}}$$

7. $$(\ln x)' = \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0}\frac{\ln(1+\frac{h}{x})}{h} = \lim_{h\to 0}\frac{\ln(1+\frac{h}{x})}{\frac{h}{x}}\cdot\frac{1}{x}$$

$$= \frac{1}{x}$$

8. $$(\log_a x)' = \lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h} = \lim_{h\to 0}\frac{\log_a(1+\frac{h}{x})}{h} = \lim_{h\to 0}\frac{\ln(1+\frac{h}{x})}{\ln a\cdot\frac{h}{x}\cdot x}$$

$$= \frac{1}{x\ln a}\lim_{h\to 0}\frac{\ln(1+\frac{h}{x})}{\frac{h}{x}} = \frac{1}{x\ln a}$$

9. $$(x^n)' = \lim_{h\to 0}\frac{(x+h)^n-x^n}{h} = x^n\lim_{h\to 0}\frac{(1+\frac{h}{x})^n-1}{h} \qquad \frac{t=(1+\frac{h}{x})^n-1}{(1+\frac{h}{x})^n=t+1} \qquad \left(\frac{\ln(1+t)}{t}\to 1\right)$$

$n\in\mathbb{R}$

$$= x^n\lim_{h\to 0}\frac{(1+\frac{h}{x})^n-1}{\ln(1+\frac{h}{x})^n}\cdot\frac{\ln(1+\frac{h}{x})^n}{h}$$

$$= x^n\lim_{t\to 0}\frac{t}{\ln(1+t)}\lim_{h\to 0}\frac{n\ln(1+\frac{h}{x})}{\frac{h}{x}\cdot x}$$

$$= x^n\lim_{t\to 0}\frac{t}{\ln(1+t)}\cdot\frac{n}{x}\lim_{h\to 0}\frac{\ln(1+\frac{h}{x})}{\frac{h}{x}}$$

$$= x^n\cdot\frac{n}{x} = nx^{n-1}$$

10. $$c' = \lim_{h\to 0}\frac{c-c}{h} = 0$$

总结：
① $c' = 0$
② $(x^n)' = nx^{n-1}$
③ [illegible]

⑤ $(e^x)' = e^x$
⑥ $(a^x)' = a^x\ln a$
⑦ $(\ln x)' = \frac{1}{x}$

③ $(\sin x)' = \cos x$

④ $(\cos x)' = -\sin x$

⑦ $(\ln x)' = \frac{1}{x}$

⑧ $(\log_a x)' = \frac{1}{x\ln a}$