多元复合函数的求导法则：f, g, h 可微

1. 一般情形：$z=f(u,v)$，$u=g(x,y)$，$v=h(x,y)$

$$\Rightarrow \frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=f_u\cdot g_x+f_v\cdot h_x$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=f_u\cdot g_y+f_v\cdot h_y$$

2. $z=f(u,v)$　$u=g(t)$，$v=h(t)$

$$\frac{dz}{dt}=\frac{\partial f}{\partial u}\cdot\frac{du}{dt}+\frac{\partial f}{\partial v}\cdot\frac{dv}{dt}=f_u\cdot g'(t)+f_v\cdot h'(t)$$

3. $z=f(t,x,y)$，$x=g(t)$，$y=h(t)$

$$\frac{dz}{dt}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

4. $w=f(x,y,z)$，$z=g(x,y)$

$$\frac{\partial w}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}=f_x+f_z\cdot g_x$$

$$\frac{\partial w}{\partial y}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}=f_y+f_z\cdot g_y$$

证明 2：$\frac{dz}{dt}=\lim\limits_{\Delta t\to 0}\frac{\Delta z}{\Delta t}$，　$z=f(u,v)$，$u=g(t)$，$v=h(t)$

$$\Delta z=f_u\cdot\Delta u+f_v\,\Delta v+o(\rho)\qquad \rho=\sqrt{\Delta u^2+\Delta v^2}$$

$$\frac{\Delta z}{\Delta t}=f_u\frac{\Delta u}{\Delta t}+f_v\frac{\Delta v}{\Delta t}+\frac{o(\rho)}{\Delta t}$$

$\Delta t\to 0$　$$\lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t}=\lim_{\Delta t\to 0}\left(\frac{f_u\cdot\Delta u}{\Delta t}+\frac{f_v\cdot\Delta v}{\Delta t}+\underbrace{\frac{o(\rho)}{\Delta t}}_{\to 0}\right)$$

$$=f_u\cdot\frac{du}{dt}+f_v\frac{dv}{dt}$$

$$\Rightarrow\quad\frac{dz}{dt}=\frac{\partial f}{\partial u}\cdot\frac{du}{dt}+\frac{\partial f}{\partial v}\cdot\frac{dv}{dt}$$

$$o(\rho)=o\left(\sqrt{\Delta u^2+\Delta v^2}\right)$$

$$\lim_{\Delta u\to 0}\frac{o(\rho)}{\Delta u}=0$$

$$\lim_{\Delta v\to 0}\frac{o(\rho)}{\Delta v}=0\Rightarrow\quad\frac{o(\rho)}{\Delta t}$$

$$\lim_{\rho\to 0}\frac{o(\rho)}{\rho}=0$$

$$\frac{o(\rho)}{\Delta t}=\frac{o(\rho)/\Delta u}{\Delta t/\Delta u}\to 0$$

例 1. $z=e^{u^2+2uv-v^2}$，$u=2x-y$，$v=x+2y$

求 $\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = (2u+2v)\, e^{u^2+2uv-v^2}\cdot 2 + (2u-2v)\, e^{u^2+2uv-v^2}\cdot 1$$

$$= e^{u^2+2uv-v^2}(4u+4v+2u-2v)$$

$$= (6u+2v)\, e^{u^2+2uv-v^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = (2u+2v)\, e^{u^2+2uv-v^2}\cdot(-1) + (2u-2v)\, e^{u^2+2uv-v^2}\cdot 2$$

$$= e^{u^2+2uv-v^2}(-2u-2v+4u-4v)$$

$$= (2u-6v)\, e^{u^2+2uv-v^2}$$

例2. $z = \sin(xy) + e^{x+2y}$ $\quad x = \sin t,\ y = \cos t$ 求 $\frac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = y\cdot\cos(xy)\cdot\cos t + x\cos(xy)(-\sin t)$$

$$= \cos(xy)(\cos^2 t - \sin^2 t) = \cos 2t\cdot\cos(xy)$$

例3. $z = \ln(t+x^2+y^2)$ $\quad x = \sin t,\ y = \cos t$

$$\frac{dz}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{1}{t+x^2+y^2} + \frac{2x}{t+x^2+y^2}\cdot\cos t + \frac{2y}{t+x^2+y^2}(-\sin t)$$

$$= \frac{1}{t+x^2+y^2}\left(1 + \underline{2\sin t\cos t} - 2\,\underline{\cos t\sin t}\right)$$

$$= \frac{1}{t+x^2+y^2}$$

例4. $w = e^{x^2+y^2+z^2}$ $\quad z = \ln(x^2+2xy-y^2)$

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = 2x\cdot e^{x^2+y^2+z^2} + 2z\, e^{x^2+y^2+z^2}\cdot\frac{2x+2y}{x^2+2xy-y^2}$$

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y} = 2y\, e^{x^2+y^2+z^2} + 2z\, e^{x^2+y^2+z^2}\cdot\frac{2x-2y}{x^2+2xy-y^2}$$