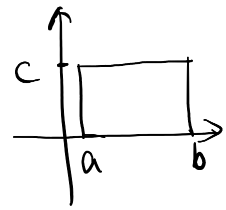
# 定积分的性质

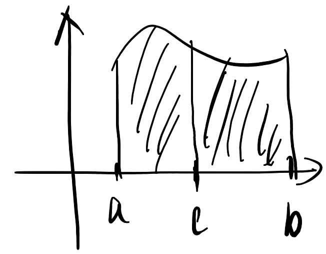
1. $\int_a^b c\,dx = c(b-a)$




2. $\int_a^b [pf(x)\pm qg(x)]dx = p\int_a^b f(x)dx \pm q\int_a^b g(x)dx$

3. (区域可加性) $c\in(a,b) \Rightarrow$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$




例1. 求定积分 $\int_0^1 (4+3x^2)dx$

$$\int_0^1 (4+3x^2)dx = \int_0^1 4\,dx + 3\int_0^1 x^2dx \qquad \left(\int_0^1 x^2dx = \frac{1}{3}\right)$$

$$= 4\cdot 1 + 3\cdot\frac{1}{3} = 5$$

例2. 已知 $\int_0^{10} f(x)dx = 17$，$\int_0^8 f(x)dx = 12$，求 $\int_8^{10} f(x)dx$.

解：$\int_0^{10} f(x)dx = \int_0^8 f(x)dx + \int_8^{10} f(x)dx$

$$\Rightarrow 17 = 12 + \int_8^{10} f(x)dx \Rightarrow \int_8^{10} f(x)dx = 5$$

性质4. 若 $f(x)\geq 0$，$\forall x\in[a,b] \Rightarrow \int_a^b f(x)dx \geq 0$

性质5. 若 $f(x)\leq g(x)$ $\forall x\in[a,b] \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$

证明：$\int_a^b g(x)dx - \int_a^b f(x)dx = \int_a^b [g(x)-f(x)]dx \geq 0$

$$\Rightarrow \int_a^b g(x)dx \geq \int_a^b f(x)dx$$

(估值定理)

性质6. 若 $m\leq f(x)\leq M \Rightarrow m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

证明：$\int_a^b m\,dx \leq \int_a^b f(x)dx \leq \int_a^b M\,dx$

$$\Rightarrow m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

性质7.(积分中值定理) $f(x)$ 在 $[a,b]$ 上连续 $\Rightarrow$ 存在 $\xi\in(a,b)$，使得

$$\int_a^b f(x)dx = f(\xi)(b-a)$$

证明. $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

$\Rightarrow \quad m \le \underbrace{\frac{1}{b-a}\int_a^b f(x)dx}_{} \le M$

$\Rightarrow$ 记 $C = \frac{1}{b-a}\int_a^b f(x)dx$, $C \in (m, M)$

由介值定理: 存在 $\xi \in (a,b)$, 使得 $f(\xi) = C$

$\Rightarrow \quad f(\xi) = \frac{1}{b-a}\int_a^b f(x)dx \quad \Rightarrow \int_a^b f(x)dx = f(\xi)(b-a)$

例3. 估计积分 $\int_1^4 \sqrt{x}\,dx$ 的值.

解. 在 $[1,4]$ 区间上, $M=2$, $m=1$

$\Rightarrow \quad 1\cdot(4-1) \le \int_1^4 \sqrt{x}\,dx \le 2\cdot(4-1)$

$\Rightarrow \quad 3 \le \int_1^4 \sqrt{x}\,dx \le 6$