方阵及其行列式

1. $A_{n\times n}$：方阵

2. 单位矩阵（单位阵）：I，$I_{n\times n}$：主对角元素全为1，其余元素全为0

$$I=\begin{pmatrix}1&&\\&\ddots&\\&&1\end{pmatrix}$$

性质：$A_{m\times n}\cdot I_{n\times n}=A_{m\times n}$

$I_{m\times m}\cdot B_{m\times n}=B_{m\times n}$

3. 对角矩阵 $A=diag(a_1,\cdots,a_n)=\begin{pmatrix}a_1&&&\\&a_2&&\\&&\ddots&\\&&&a_n\end{pmatrix}$

除主对角线外，其余元素全为0的方阵

4. 对称矩阵：$A^T=A$，$a_{ij}=a_{ji}$

反对称矩阵：$A^T=-A$，$a_{ij}=-a_{ji}$

5. 方阵的幂 $A^n=\underbrace{A\cdot A\cdots\cdot A}_{n个}$，$\underline{A^0=I}$

6. 方阵的行列式：$\det A=|A|$

性质：① $|A^T|=|A|$

② $|\lambda A|=\lambda^n|A|$

③ $|AB|=|A||B|$

证明③：用二阶方阵来证明，三阶以上的证明类似。

考虑 $M=\begin{pmatrix}A&0\\-I&B\end{pmatrix}=\begin{pmatrix}a_{11}&a_{12}&0&0\\a_{21}&a_{22}&0&0\\-1&0&b_{11}&b_{12}\\0&-1&b_{21}&b_{22}\end{pmatrix}$

$|M|=|A||B|$ 展开行列式即可

$$|M| = \begin{vmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ -1 & 0 & b_{11} & b_{12} \\ 0 & -1 & b_{21} & b_{22} \end{vmatrix} = \begin{vmatrix} 0 & a_{12} & a_{11}b_{11} & a_{11}b_{12} \\ a_{21} & a_{22} & 0 & 0 \\ -1 & 0 & b_{11} & b_{12} \\ 0 & -1 & b_{21} & b_{22} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & a_{12} & a_{11}b_{11} & a_{11}b_{12} \\ 0 & a_{22} & a_{21}b_{11} & a_{21}b_{12} \\ -1 & 0 & b_{11} & b_{12} \\ 0 & -1 & b_{21} & b_{22} \end{vmatrix} = \begin{vmatrix} 0 & 0 & a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ 0 & 0 & a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \\ -1 & 0 & b_{11} & b_{12} \\ 0 & -1 & b_{21} & b_{22} \end{vmatrix}$$

$$= \underline{(-1)(-1)}^{3+1} \begin{vmatrix} 0 & a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ 0 & a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \\ \underline{-1} & b_{21} & b_{22} \end{vmatrix}$$

$$= \underline{(-1)}(\underline{-1})(-1)^{3+1} \begin{vmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{vmatrix}$$

$$= \begin{vmatrix} \underline{a_{11}b_{11}+a_{12}b_{21}} & a_{11}\underline{b_{12}+a_{12}b_{22}} \\ a_{21}\underline{b_{11}+a_{22}b_{21}} & a_{21}\underline{b_{12}+a_{22}}b_{22} \end{vmatrix}$$

$$= |AB|$$