

齐次方程:

1. 齐次方程:  $y' = f(\frac{y}{x})$

做代换:  $u = \frac{y}{x} \Rightarrow y = xu. \Rightarrow \frac{dy}{dx} = u + x \frac{du}{dx}$

$$\Rightarrow u + x \frac{du}{dx} = f(u) \Rightarrow x \frac{du}{dx} = f(u) - u.$$

$$\Rightarrow \frac{du}{f(u) - u} = \frac{dx}{x}$$

例1. 解方程:  $y' = \frac{y + x e^{-y/x}}{x}$

解:  $y' = \frac{y}{x} + e^{-y/x}$ , 令  $u = \frac{y}{x} \Rightarrow \frac{dy}{dx} = u + x \frac{du}{dx}$

$$\Rightarrow u + x \frac{du}{dx} = u + e^{-u} \Rightarrow x \frac{du}{dx} = e^{-u} \Rightarrow e^u du = \frac{dx}{x}$$

$$\Rightarrow e^u = \ln|x| + C$$

$$\Rightarrow u = \ln(\ln|x| + C)$$

$$\Rightarrow \frac{y}{x} = \ln(\ln|x| + C) \Rightarrow y = x \ln(\ln|x| + C)$$

例2. 解方程  $x^2 y' = y^2 + xy - x^2$

解:  $x^2 y' = y^2 + xy - x^2 \Rightarrow y' = (\frac{y}{x})^2 + \frac{y}{x} - 1$

$$\Rightarrow u = \frac{y}{x} \Rightarrow u + x \frac{du}{dx} = u^2 + u - 1 \Rightarrow x \frac{du}{dx} = u^2 - 1$$

$$\Rightarrow \frac{du}{u^2 - 1} = \frac{dx}{x} \Rightarrow \int \frac{du}{u^2 - 1} = \int \frac{dx}{x} = \ln|x| + C$$

$$\frac{1}{u^2 - 1} = \frac{A}{u - 1} + \frac{B}{u + 1}$$

$$\Rightarrow A(u + 1) + B(u - 1) = 1$$

$$\Rightarrow u = 1 \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$$

$$u = -1 \Rightarrow -2B = 1 \Rightarrow B = -\frac{1}{2}$$

$$\Rightarrow \int \frac{du}{u^2-1} = \frac{1}{2} \int \frac{du}{u-1} - \frac{1}{2} \int \frac{du}{u+1}$$

$$= \frac{1}{2} \ln|u-1| - \frac{1}{2} \ln|u+1| = \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right|$$

$$\Rightarrow \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| = \ln|x| + C \quad \frac{u-1}{u+1} = \frac{\frac{y}{x}-1}{\frac{y}{x}+1} = \frac{y-x}{y+x}$$

$$\Rightarrow \ln \left| \frac{u-1}{u+1} \right|^{\frac{1}{2}} = \ln|x| + C$$

$$\Rightarrow \ln \left| \frac{y-x}{y+x} \right|^{\frac{1}{2}} = \ln|x| + C$$

$$= \ln|cx|$$

$$C = \ln c$$

$$\Rightarrow \left| \frac{y-x}{y+x} \right|^{\frac{1}{2}} = |cx| \quad \Rightarrow \quad \frac{y-x}{y+x} = cx^2$$

$$\Rightarrow y-x = cx^2(y+x)$$

$$\Rightarrow y(1-cx^2) = cx^3+x$$

$$\Rightarrow y = \frac{x(1+cx^2)}{1-cx^2}$$