基变换与坐标变换

$$\vec{\alpha}=\begin{pmatrix}7\\3\\1\end{pmatrix}=7\vec{e_1}+3\vec{e_2}+\vec{e_3}$$
$$=\vec{\alpha_1}-2\vec{\alpha_2}+6\vec{\alpha_3}$$

$$(\vec{\alpha_1},\vec{\alpha_2},\vec{\alpha_3})=(\vec{e_1},\vec{e_2},\vec{e_3})(\vec{\alpha_1},\vec{\alpha_2},\alpha_3)$$

1. 基变换：设线性空间 V 有两组基 $\vec{\alpha_1},\cdots,\vec{\alpha_n}$ 和 $\vec{\beta_1},\cdots,\vec{\beta_n}$，

$$\begin{cases}\vec{\beta_1}=k_{11}\vec{\alpha_1}+k_{21}\vec{\alpha_2}+\cdots+k_{n1}\vec{\alpha_n}=(\vec{\alpha_1}\cdots\vec{\alpha_n})\begin{pmatrix}k_{11}\\k_{21}\\\vdots\\k_{n1}\end{pmatrix}\\ \vec{\beta_2}=k_{12}\vec{\alpha_1}+k_{22}\vec{\alpha_2}+\cdots+k_{n2}\vec{\alpha_n}=(\vec{\alpha_1}\cdots\vec{\alpha_n})\begin{pmatrix}k_{12}\\k_{22}\\\vdots\\k_{n2}\end{pmatrix}\\ \vdots\\ \vec{\beta_n}=k_{1n}\vec{\alpha_1}+k_{2n}\vec{\alpha_2}+\cdots+k_{nn}\vec{\alpha_n}=(\vec{\alpha_1}\cdots\vec{\alpha_n})\begin{pmatrix}k_{1n}\\\vdots\\k_{nn}\end{pmatrix}\end{cases}$$

$$\Rightarrow (\vec{\beta_1}\ \vec{\beta_2}\cdots\vec{\beta_n})=(\vec{\alpha_1}\cdots\vec{\alpha_n})\begin{pmatrix}k_{11}&\cdots&k_{1n}\\\vdots&&\vdots\\k_{n1}&\cdots&k_{nn}\end{pmatrix}=(\vec{\alpha_1}\cdots\vec{\alpha_n})P$$

$\Rightarrow$ P 称之为从基 $\vec{\alpha_1},\cdots,\alpha_n$ 到 $\vec{\beta_1},\cdots,\vec{\beta_n}$ 的过渡矩阵.

$$\Rightarrow (\vec{\alpha_1},\cdots,\vec{\alpha_n})=(\vec{\beta_1},\cdots,\vec{\beta_n})P^{-1}$$

2. 坐标变换：

$$\vec{x}=a_1\vec{\alpha_1}+\cdots+a_n\vec{\alpha_n}=(\vec{\alpha_1}\cdots\alpha_n)\begin{pmatrix}a_1\\\vdots\\a_n\end{pmatrix}=(\vec{\beta_1},\cdots,\vec{\beta_n})P^{-1}\begin{pmatrix}a_1\\\vdots\\a_n\end{pmatrix}$$
$$=b_1\vec{\beta_1}+\cdots+b_n\vec{\beta_n}=(\vec{\beta_1}\cdots\vec{\beta_n})\begin{pmatrix}b_1\\\vdots\\b_n\end{pmatrix}=(\vec{\alpha_1},\cdots,\alpha_n)P\begin{pmatrix}b_1\\\vdots\\b_n\end{pmatrix}$$

$$\because (\vec{\beta_1},\cdots,\beta_n)=(\vec{\alpha_1},\cdots,\vec{\alpha_n})P \Rightarrow$$

$$\Rightarrow \begin{pmatrix}a_1\\\vdots\\a_n\end{pmatrix}=P\begin{pmatrix}b_1\\\vdots\\b_n\end{pmatrix},\quad \begin{pmatrix}b_1\\\vdots\\b_n\end{pmatrix}=P^{-1}\begin{pmatrix}a_1\\\vdots\\a_n\end{pmatrix}$$

例1. 在 $P_3[x]$ 中取两个基

$\vec{\alpha}_1 = x^3+2x^2-x$, $\vec{\alpha}_2 = x^3-x^2+x+1$, $\vec{\alpha}_3 = -x^3+2x^2+x+1$, $\vec{\alpha}_4 = -x^3-x^2+1$

$\vec{\beta}_1 = 2x^3+x^2+1$, $\vec{\beta}_2 = x^2+2x+2$, $\vec{\beta}_3 = -2x^3+x^2+x+2$, $\vec{\beta}_4 = x^3+3x^2+x+2$.

求坐标变换公式.

解:
$$(\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3, \vec{\alpha}_4) = (x^3, x^2, x, 1)\underbrace{\begin{pmatrix} 1 & 1 & -1 & -1 \\ 2 & -1 & 2 & -1 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}}_{A} = (x^3, x^2, x, 1)A$$

$$(\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3, \vec{\beta}_4) = (x^3, x^2, x, 1)\begin{pmatrix} 2 & 0 & -2 & 1 \\ 1 & 1 & 1 & 3 \\ 0 & 2 & 1 & 1 \\ 1 & 2 & 2 & 2 \end{pmatrix} = (x^3, x^2, x, 1)B.$$

$$\Rightarrow (x^3, x^2, x, 1) = (\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3, \vec{\alpha}_4)A^{-1}$$

$$\Rightarrow (\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3, \vec{\beta}_4) = (\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3, \vec{\alpha}_4)A^{-1}B \qquad \Rightarrow P = A^{-1}B$$

$$\Rightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = A^{-1}B\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = B^{-1}A\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \qquad P^{-1} = B^{-1}A$$

现在求 $B^{-1}A$. 做法: $(B \mid A) \sim (E \mid B^{-1}A)$, $(A \mid B) \sim (E \mid A^{-1}B)$

$$B^{-1}A = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$