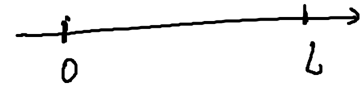
稳定态，极小曲面和拉普拉斯方程。

1. 稳定态：与时间 t 无关，$\Rightarrow u_t=0,\ u_{tt}=0$

热传导方程：$u_t=c^2\Delta u$ $\quad u_t=0 \Rightarrow \Delta u=0$

一维：$u_t=c^2u_{xx}$ $\quad u_t=0 \Rightarrow u_{xx}=0 \Rightarrow \underline{u=A\underline{x}+B}$



波动方程：$u_{tt}=c^2\Delta u$，$\quad u_{tt}=0 \Rightarrow \Delta u=0$

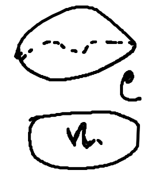
一维：$u_{tt}=c^2u_{xx}$，$\quad u_{tt}=0 \Rightarrow u=Ax+B$

$\Delta u=0$：拉普拉斯方程。 Δ：拉普拉斯算子。

n维：$\Delta u=\dfrac{\partial^2 u}{\partial x_1^2}+\dfrac{\partial^2 u}{\partial x_2^2}+\cdots+\dfrac{\partial^2 u}{\partial x_n^2}$.

三维：$\Delta u=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}$

2. 极小曲面：以 C 为边界的曲面中，哪个曲面面积最小？曲面方程 $v=v(x,y)$



$C:\begin{cases} x=x(s) \\ y=y(s) \\ v=\varphi(s) \end{cases}$ 曲面投影区域为 Ω

曲面的面积：$A=\iint_\Omega \sqrt{v_x^2+v_y^2+1}\,dA$

记 $\underline{M(\varphi)}=\{v \mid v|_{\partial\Omega}=\varphi\}$.

$\underline{J(v)}=\iint_\Omega \sqrt{v_x^2+v_y^2+1}\,dA$ （v 的面积）

（泛函：以函数为自变量的函数）求 $J(v)$ 的极小值（泛函的极值问题，变分法）

① 假设 $J(u)$ 是 $J(v)$ 的极小值，将 u 做微小扰动 $\underline{u+\varepsilon v}$.

其中 $v\in M_0(v)=\{v \mid v|_{\partial\Omega}=0\} \Rightarrow u+\varepsilon v\in M(\varphi)$

记 $J(\varepsilon)=J(u+\varepsilon v)=\iint_\Omega \sqrt{(u_x+\varepsilon v_x)^2+(u_y+\varepsilon v_y)^2+1}\,dA$

若 u 是极小值，则在 $\varepsilon=0$ 时，$\underline{J'(\varepsilon)=0}$ （第一变分）

$$J'(\varepsilon)=\iint_\Omega \frac{\cancel{2}(u_x+\varepsilon v_x)\cdot v_x+\cancel{2}(u_y+\varepsilon v_y)\cdot v_y}{\cancel{2}\sqrt{(u_x+\varepsilon v_x)^2+(u_y+\varepsilon v_y)^2+1}}\,dA.$$

$$=\iint_\Omega \frac{(u_x+\varepsilon v_x)\cdot v_x+(u_y+\varepsilon v_y)v_y}{\sqrt{(u_x+\varepsilon v_x)^2+(u_y+\varepsilon v_y)^2+1}}\,dA.$$

$$\underline{\underline{J'(0)}}=\iint_\Omega \frac{u_x\cdot v_x+u_y\cdot v_y}{\sqrt{u_x^2+u_y^2+1}}\,dA=\iint_\Omega \frac{1}{\sqrt{u_x^2+u_y^2+1}}\nabla u\cdot\nabla v\,dA$$

格林公式

$$=\oint_{\underline{\partial\Omega}}\frac{1}{\sqrt{u_x^2+u_y^2+1}}\underline{v}\cdot\frac{\partial u}{\partial \vec{n}}\,ds-\iint_\Omega v\cdot\left[\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{u_x^2+u_y^2+1}}\right)+\frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{u_x^2+u_y^2+1}}\right)\right]dA$$

$v|_{\partial\Omega}=0$
$$=-\iint_\Omega v\cdot\underbrace{\left[\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{u_x^2+u_y^2+1}}\right)+\frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{u_x^2+u_y^2+1}}\right)\right]}\,dA.=0$$

$\because v\in M_0(v)$ 任意 $\Rightarrow$
$$\Rightarrow\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{u_x^2+u_y^2+1}}\right)+\frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{u_x^2+u_y^2+1}}\right)=0$$

（极小曲面方程，非线性方程）

$$J''(\varepsilon)=\frac{d}{d\varepsilon}\iint_\Omega\frac{(u_x+\underline{\varepsilon v_x})\cdot v_x+(u_y+\underline{\varepsilon v_y})v_y}{\sqrt{1+(u_x+\varepsilon v_x)^2+(u_y+\varepsilon v_y)^2}}\,dA$$

$$=\iint_\Omega\frac{(v_x^2+v_y^2)\sqrt{(u_x+\varepsilon v_x)^2+(u_y+\varepsilon v_y)^2+1}-\left[(u_x+\varepsilon v_x)v_x+(u_y+\varepsilon v_y)v_y\right]\frac{(u_y+\varepsilon v_y)v_y+(u_x+\varepsilon v_x)v_x}{\sqrt{(u_x+\varepsilon v_x)^2+\cdots}}}{1+(u_x+\varepsilon v_x)^2+(u_y+\varepsilon v_y)^2}$$

$$=\iint_\Omega\frac{v_x^2+v_y^2+\left[v_x(u_y+\varepsilon v_y)-v_y(u_x+\varepsilon v_x)\right]^2}{\left(1+(u_x+\varepsilon v_x)^2+(u_y+\varepsilon v_y)^2\right)^{3/2}}\,dA\geq 0$$

$\Rightarrow J(0)$ 是极小值.

$$\Rightarrow \text{极小曲面 } u:\ \frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{u_x^2+u_y^2+1}}\right)+\frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{u_x^2+u_y^2+1}}\right)=0$$

再一步假定 $u_x \ll 1$，$u_y \ll 1$.（u_x, u_y 足够小）

$$\sqrt{u_x^2+u_y^2+1} \approx 1$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{u_x^2+u_y^2+1}}\right)+\frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{u_x^2+u_y^2+1}}\right)=0 \approx u_{xx}+u_{yy}=0$$

$$\Rightarrow u_{xx}+u_{yy}=0 \Rightarrow \Delta u=0 \Rightarrow \text{拉普拉斯方程.}$$