

非齐次方程, 非齐次边界条件的分离变量法

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t), & t > 0, \quad 0 < x < L \\ u(0,t) = \varphi(t), \quad u(L,t) = \psi(t), & t > 0 \\ u(x,0) = g(x) \end{cases}$$

设 $u(x,t) = v(x,t) + w(x,t)$, 其中 v, w 满足

$$\begin{cases} v_t = a^2 v_{xx} \\ v(0,t) = \varphi(t), \quad v(L,t) = \psi(t) \\ v(x,0) = g(x) \end{cases}$$

⇒ 边界条件齐次化.

$$v = v_1 + v_2, \quad v_2(0,t) = \varphi(t), \quad v_2(L,t) = \psi(t) \\ v_2 = \frac{\psi(t) - \varphi(t)}{L} \cdot x + \varphi(t)$$

$$\begin{cases} w_t = a^2 w_{xx} + f(x,t) \\ w(0,t) = 0, \quad w(L,t) = 0 \\ w(x,0) = 0 \end{cases}$$

⇒ 特征函数展开法.

$$w(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin \frac{n\pi x}{L} \Rightarrow \text{代入方程, 求出 } A_n(t)$$

$$\Rightarrow u = v + w$$